

Bosonic symmetries of the extended fermionic $(2N, 2M)$ -Toda hierarchy

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Abstract

In this paper, we construct the additional symmetries of the fermionic $(2N, 2M)$ -Toda hierarchy basing on the generalization of the $N=(1|1)$ supersymmetric two dimensional Toda lattice hierarchy. These additional flows constitute a $w_\infty \times w_\infty$ Lie algebra. As a Bosonic reduction of the $N=(1|1)$ supersymmetric two dimensional Toda lattice hierarchy and the fermionic $(2N, 2M)$ -Toda hierarchy, we define a new extended fermionic $(2N, 2M)$ -Toda hierarchy which admits a Bosonic Block type superconformal structure.

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1 Introduction

In the study of integrable hierarchies, it is interesting to identify their symmetries and the algebraic structure of the symmetries, particularly to find the additional symmetry. Additional symmetries of the Kadomtsev-Petviashvili(KP) hierarchy were introduced by Orlov and Shulman [1] which contain one important symmetry called Virasoro symmetry. As a nonlinear evolutionary differential-difference

equation describing an infinite system of masses on a line that interact through an exponential force, the Toda equation[2] was generalized to Toda lattice hierarchy [3] which is completely integrable and has important applications in many fields such as classical and quantum field theory, in particular in the theory of Gromov-Witten invariants ([4]). Adding extended logarithmic flows, the Toda lattice hierarchy was extended into the so-called extended Toda hierarchy ([5]) and was conjectured and shown ([6], [7], [4]) that the extended Toda hierarchy is a hierarchy describing the Gromov-Witten invariants of CP^1 as the large N limit of the CP^1 topological sigma model. The extended bigraded Toda hierarchy (EBTH) was introduced by Guido Carlet ([8]) who hoped that EBTH might also be relevant for some applications in two dimensional(2D) topological field theory and in the theory of Gromov-Witten invariants. The Hirota bilinear equation of EBTH were equivalently constructed in our early paper [9] and a very recent paper [10], because of the equivalence of $t_{1,N}$ flow and $t_{0,N}$ flow of the EBTH in [9]. Meanwhile it was proved to govern Gromov-Witten invariant of the total descendent potential of \mathbb{P}^1 orbifolds [10]. This hierarchy also attracted a series of results from analytical and algebraic considerations [11, 12, 13, 14, 15].

Various generalizations and supersymmetric extensions [16] of the KP hierarchy have deep implications in mathematical physics, particularly in the theory of Lie algebras. One important supersymmetric extension is the supersymmetric Manin-Radul Kadomtsev-Petviashvili (MR-SKP) hierarchy[17] which contains a lot of reduced integrable super solitary equations including the Kupershmidt's super-KdV equation. The additional symmetry of the MR-SKP hierarchy was studied by Stanciu [18]. Later the supersymmetric BKP (SBKP) hierarchy was constructed in [19]. In our quite recent paper[20], we constructed the additional symmetries of the supersymmetric BKP hierarchy. These additional flows constitute a B type $SW_{1+\infty}$ Lie algebra. Further we generalize the SBKP hierarchy to a supersymmetric two-component BKP hierarchy (S2BKP) hierarchy and derive its algebraic structure. As a reduction of the S2BKP hierarchy, a new supersymmetric Drinfeld-Sokolov hierarchy of type D was constructed and proved to have a super Block type additional symmetry. Later the Darboux transformation of the supersymmetric BKP (SBKP) hierarchy was constructed in [21].

The supersymmetric Toda hierarchy was introduced in [22]. By introducing the Lie superalgebra $osp(\infty|\infty)$, the ortho-symplectic supersymmetric Toda hierarchy was defined as well in [23]. These equations in the hierarchy were solved through the Riemann-Hilbert decomposition of corresponding infinite dimensional Lie supergroups. Recently, the supersymmetric Toda hierarchy was studied a lot including their hamiltonian structure, dispersionless limit and so on [24]-[27]. Recently, about the application of Block algebra in KP and Toda type systems, we also prove that the constrained BKP system and Toda system all have a Block symmetry in [28, 29].

This paper is arranged as follows. In the next section, some fundamental notations on the space of supersymmetric shift operators will be prepared. After this, we introduce some necessary facts of the $N=(1|1)$ supersymmetric Toda hierarchy in Section 3. In Sections 4, we will give an introduction of the fermionic (K, S) -Toda hierarchy and we will give the additional symmetries for the fermionic $(2N, 2M)$ -Toda hierarchy and derive its $w_\infty \times w_\infty$ symmetry in Section 5. As a Bosonic reduction of the fermionic $(2N, 2M)$ -Toda hierarchy, we define a new constrained system called the extended fermionic $(2N, 2M)$ -Toda hierarchy in Section 6 and prove that it possesses a Bosonic Block type Lie symmetry in Section 7.

2 Supersymmetric shift operators

In this section, we will recall an introduction about the space of supersymmetric shift operators following a series of papers of Sorin's group, for example the reference [27]. Firstly the space of difference operators can be represented in the following general form

$$\mathbb{F}_m = \sum_{k=-\infty}^{\infty} f_k^{(m)}(x) \Lambda^{k-m}, \quad \Lambda = e^{\epsilon \partial}, \quad m, k \in \mathbb{Z}, \quad (2.1)$$

where the parameter ϵ in the class of difference operators is the string coupling constant and Λ is the shift operator whose action on the continuous spatial fields as a shift of a lattice index ϵ

$$\Lambda^l f_k^{(m)}(x) = f_k^{(m)}(x + l\epsilon) \Lambda^l. \quad (2.2)$$

The functions $f_{2k}(x)^{(m)}$ ($f_{2k+1}(x)^{(m)}$) are the bosonic (fermionic) lattice fields and the Grassmann Z_2 -parity can be defined

$$d_{f_k^{(m)}(x)} = |k| \bmod 2, \quad d_{\Lambda^l} = |l| \bmod 2.$$

The operators \mathbb{F}_m in (2.1) admit the diagonal Z_2 -parity

$$d_{\mathbb{F}_m} = d_{f_k(x)^{(m)}} + d_{\Lambda^{k-m}} = |m| \bmod 2. \quad (2.3)$$

The involution $*$ can be defined as [22]

$$\mathbb{F}_m^* = \sum_{k=-\infty}^{\infty} (-1)^k f_k^{(m)}(x) \Lambda^{k-m}. \quad (2.4)$$

In the follows we need the projections of the operators \mathbb{F}_m defined as

$$(\mathbb{F}_m)_{\leq p} = \sum_{k \leq p+m} f_k(x) \Lambda^{k-m}, \quad (\mathbb{F}_m)_{\geq p} = \sum_{k \geq p+m} f_k(x) \Lambda^{k-m},$$

and we will use the usual notation for the projections $(\mathbb{F}_m)_+ := (\mathbb{F}_m)_{\geq 0}$ and $(\mathbb{F}_m)_- := (\mathbb{F}_m)_{< 0}$.

The generalized graded algebra on these subspaces will be defined with the following bracket [27]

$$[\mathbb{F}, \widetilde{\mathbb{F}}] := \mathbb{F} \widetilde{\mathbb{F}} - (-1)^{d_{\mathbb{F}} d_{\widetilde{\mathbb{F}}}} \widetilde{\mathbb{F}}^{*(d_{\mathbb{F}})} \mathbb{F}^{*(d_{\widetilde{\mathbb{F}}})}, \quad (2.5)$$

where $\mathbb{F}^{*(j)}$ denotes the j -fold action of the involution $*$ on the operator \mathbb{F} with $\mathbb{F}^{*(2)} = \mathbb{F}$. The bracket (2.5) generalizes the (anti)commutator in superalgebras and satisfies the following properties [27] including symmetry, derivation and Jacobi identity as

$$[\mathbb{F}, \widetilde{\mathbb{F}}] = -(-1)^{d_{\mathbb{F}} d_{\widetilde{\mathbb{F}}}} [\widetilde{\mathbb{F}}^{*(d_{\mathbb{F}})}, \mathbb{F}^{*(d_{\widetilde{\mathbb{F}}})}], \quad (2.6)$$

$$[\mathbb{F}, \widetilde{\mathbb{F}} \widehat{\mathbb{F}}] = [\mathbb{F}, \widetilde{\mathbb{F}}] \widehat{\mathbb{F}} + (-1)^{d_{\mathbb{F}} d_{\widetilde{\mathbb{F}}}} \widetilde{\mathbb{F}}^{*(d_{\mathbb{F}})} [\mathbb{F}^{*(d_{\widetilde{\mathbb{F}}})}, \widehat{\mathbb{F}}], \quad (2.7)$$

$$\begin{aligned}
& (-1)^{d_{\mathbb{F}} d_{\widetilde{\mathbb{F}}}} [[\mathbb{F}, \widetilde{\mathbb{F}}^*(d_{\mathbb{F}})], \widehat{\mathbb{R}}^*(d_{\mathbb{F}}+d_{\widetilde{\mathbb{F}}})] + (-1)^{d_{\widetilde{\mathbb{F}}} d_{\mathbb{F}}} [[\widetilde{\mathbb{F}}, \widehat{\mathbb{R}}^*(d_{\widetilde{\mathbb{F}}})], \mathbb{R}^*(d_{\mathbb{F}}+d_{\widetilde{\mathbb{F}}})] \\
& + (-1)^{d_{\widetilde{\mathbb{F}}} d_{\mathbb{F}}} [[\widehat{\mathbb{R}}, \mathbb{R}^*(d_{\widetilde{\mathbb{F}}})], \widetilde{\mathbb{R}}^*(d_{\mathbb{F}}+d_{\widetilde{\mathbb{F}}})] = 0.
\end{aligned} \tag{2.8}$$

Same as [22], the operator \mathbb{R}_*^k is defined as following

$$\mathbb{R}_*^{2k} = (\mathbb{R}^*(d_{\mathbb{F}})\mathbb{F})^k, \quad \mathbb{R}_*^{2k+1} = \mathbb{F}\mathbb{R}_*^{2k}. \tag{2.9}$$

With the above preparation, the $N=(1|1)$ supersymmetric 2D Toda lattice hierarchy will be introduced as [26] in the next section.

3 $N=(1|1)$ supersymmetric 2D Toda lattice hierarchy

The Lax operators L, \bar{L} of the $N=(1|1)$ supersymmetric 2D Toda lattice hierarchy belong to the space of operators

$$L = \sum_{k=0}^{\infty} u_k(x) \Lambda^{1-k}, \quad u_0(x) = 1, \quad \bar{L} = \sum_{k=0}^{\infty} v_k(x) \Lambda^{k-1}, \quad v_0(x) \neq 0 \tag{3.1}$$

and have the grading $d_L = d_{\bar{L}} = 1$. The coefficient functions in the lax operators depend on infinitely many dependent variables. Then the Lax operators L, \bar{L} can have the following dressing structures

$$L = W^* \Lambda W^{-1}, \quad \bar{L} = \bar{W}^* \Lambda^{-1} \bar{W}^{-1}, \tag{3.2}$$

where

$$W = \sum_{k=0}^{\infty} W_k(x) \Lambda^{-k}, \quad W_0(x) = 1, \quad \bar{W} = \sum_{k=0}^{\infty} \bar{W}_k(x) \Lambda^k. \tag{3.3}$$

Here $d_W = d_{\bar{W}} = 0 \bmod 2$. Then the Lax operators L_*^{2n}, \bar{L}_*^{2m} can have the following dressing structures

$$L_*^{2n} := (L^* L)^n = (W \Lambda W^{*-1} W^* \Lambda W^{-1})^n = W \Lambda^{2n} W^{-1}, \tag{3.4}$$

$$\bar{L}_*^{2m} := (\bar{L}^* \bar{L})^m = (\bar{W} \Lambda^{-1} \bar{W}^{*-1} \bar{W}^* \Lambda^{-1} \bar{W}^{-1})^m = \bar{W} \Lambda^{-2m} \bar{W}^{-1}, \tag{3.5}$$

and the Lax operators $L_*^{2n+1}, \bar{L}_*^{2m+1}$ can have the following dressing structures

$$L_*^{2n+1} := L(L^* L)^n = W^* \Lambda^{2n+1} W^{-1}, \tag{3.6}$$

$$\bar{L}_*^{2m+1} := \bar{L}(\bar{L}^* \bar{L})^m = \bar{W}^* \Lambda^{-2m-1} \bar{W}^{-1}. \tag{3.7}$$

The Lax representation of the $N=(1|1)$ supersymmetric 2D Toda lattice hierarchy in terms of bracket operations is as following [26]

$$D_n L = -(-1)^n [((L_*^n)_-)^*, L], \quad n \in \mathbb{N}, \tag{3.8}$$

$$D_n \bar{L} = (-1)^n [((L^n)_+)^*, \bar{L}], \quad n \in \mathbb{N}, \quad (3.9)$$

$$\bar{D}_n L = -(-1)^n [((\bar{L}^n)_-)^*, L], \quad n \in \mathbb{N}, \quad (3.10)$$

$$\bar{D}_n \bar{L} = (-1)^n [((\bar{L}^n)_+)^*, \bar{L}], \quad n \in \mathbb{N}, \quad (3.11)$$

where D_{2n} (D_{2n+1}) and \bar{D}_{2n} (\bar{D}_{2n+1}) are bosonic (fermionic) evolution derivatives.

The Lax equations generate a non-Abelian algebra of flows of the $N=(1|1)$ supersymmetric 2D Toda lattice hierarchy,

$$[D_n, \bar{D}_l] = [D_n, D_{2l}] = [\bar{D}_n, \bar{D}_{2l}] = 0, \quad (3.12)$$

$$\{D_{2n+1}, D_{2l+1}\} = 2D_{2(n+l+1)}, \quad \{\bar{D}_{2n+1}, \bar{D}_{2l+1}\} = 2\bar{D}_{2(n+l+1)}, \quad (3.13)$$

which can be realized as

$$D_{2n} = \partial_{2n}, \quad D_{2n+1} = \partial_{2n+1} + \sum_{l=1}^{\infty} t_{2l-1} \partial_{2(n+l)}, \quad \partial_m = \partial_{t_m}, \quad (3.14)$$

$$\bar{D}_{2n} = \bar{\partial}_{2n}, \quad \bar{D}_{2n+1} = \bar{\partial}_{2n+1} + \sum_{l=1}^{\infty} \bar{t}_{2l-1} \bar{\partial}_{2(n+l)}, \quad \bar{\partial}_m = \partial_{\bar{t}_m}, \quad (3.15)$$

where t_{2n} (t_{2n+1}), \bar{t}_{2n} (\bar{t}_{2n+1}) are bosonic (fermionic) evolution times.

Similarly to the Lax operators L, \bar{L} , the operators L_*^n, \bar{L}_*^n can be represented as

$$L_*^m := \sum_{k=0}^{\infty} u_k^{(m)}(x) \Lambda^{m-k}, \quad u_0^{(m)}(x) = 1, \quad \bar{L}_*^m := \sum_{k=0}^{\infty} v_k^{(m)}(x) \Lambda^{k-m}. \quad (3.16)$$

Here $u_k^{(m)}(x)$ and $v_k^{(m)}(x)$ are functionals of the original fields $\{u_k(x), v_k(x)\}$. The Z_2 -grading of the operator L_*^n, \bar{L}_*^n has the form $d_{L_*^{2n}} = d_{\bar{L}_*^{2n}} = 0$ and $d_{L_*^{2n+1}} = d_{\bar{L}_*^{2n+1}} = 1$. This agrees with another Z_2 -grading $d_{D_{2n}} = d_{\bar{D}_{2n}} = 0$ and $d_{D_{2n+1}} = d_{\bar{D}_{2n+1}} = 1$ that corresponds to the statistics of the evolution derivatives D_n, \bar{D}_n .

Using the definitions of L_*^n, \bar{L}_*^n , we can easily obtain the following useful identities [26]

$$\begin{aligned} [L_*^{2n}, L_*^{2m}] &= 0, \\ [(L_*^{2n})^*, L_*^{2m+1}] &= 0, \quad [L_*^{2n+1}, L_*^{2m}] = 0, \\ [(L_*^{2n+1})^*, L_*^{2m+1}] &= 2L_*^{2(n+m+1)}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} [\bar{L}_*^{2n}, \bar{L}_*^{2m}] &= 0, \\ [(\bar{L}_*^{2n})^*, \bar{L}_*^{2m+1}] &= 0, \quad [\bar{L}_*^{2n+1}, \bar{L}_*^{2m}] = 0, \\ [(\bar{L}_*^{2n+1})^*, \bar{L}_*^{2m+1}] &= 2\bar{L}_*^{2(n+m+1)}. \end{aligned} \quad (3.18)$$

Next, using (3.8–3.11), one can derive dynamical equations for the operators L_*^n, \bar{L}_*^n [26],

$$D_n L_*^m = -(-1)^{nm} [((L_*^n)_-)^{*(m)}, L_*^m], \quad (3.19)$$

$$D_n \bar{L}_*^m = (-1)^{nm} [((L_*^n)_+)^{*(m)}, \bar{L}_*^m], \quad (3.20)$$

$$\bar{D}_n L_*^m = -(-1)^{nm} [((\bar{L}_*^n)_-)^{*(m)}, L_*^m], \quad (3.21)$$

$$\bar{D}_n (\bar{L}_*)^m = (-1)^{nm} [((\bar{L}_*^n)_+)^{*(m)}, \bar{L}_*^m]. \quad (3.22)$$

The supersymmetric $N=(1|1)$ supersymmetric 2D Toda lattice equation

$$D_1 \bar{D}_1 \ln v_0(x) = v_0(x + \epsilon) - v_0(x - \epsilon) \quad (3.23)$$

belongs to system of equations (3.8–3.11).

4 2D fermionic (K, S) -Toda lattice hierarchy

In this section, we construct the two-dimensional fermionic (K, S) -Toda lattice hierarchy in terms of the Lax-pair representation similarly as the fermionic (K, S) -Toda lattice hierarchy in [26].

Let us consider two difference operators

$$L_K = L^K = \sum_{k=0}^{\infty} u_{Kk}(x) \Lambda^{K-k}, \quad \bar{L}_S = \bar{L}^S = \sum_{k=0}^{\infty} v_{Sk}(x) \Lambda^{k-S}, \quad (4.1)$$

which have the Grassmann parities as $d_{L_K} = K$, $d_{\bar{L}_S} = S$. If $K = 2N, S = 2M$, then the Lax operators L^{2N}, \bar{L}^{2M} can have the following dressing structures

$$L_K = L^{2N} := (L^* L)^N = W \Lambda^{2N} W^{-1}, \quad (4.2)$$

$$L_S = \bar{L}^{2M} := (\bar{L}^* \bar{L})^M = \bar{W} \Lambda^{-2M} \bar{W}^{-1}. \quad (4.3)$$

The dynamics of the fields $u_{Kk}(x), v_{Sk}(x)$ are governed by the Lax equations expressed in terms of the generalized graded bracket (2.5) [27]

$$\begin{aligned} D_s(L_K)^r &= -(-1)^{srK} [(((L_K)^s)_-)^{*(rK)}, (L_K)^r], \\ D_s(\bar{L}_S)^r &= (-1)^{srKS} [(((L_K)^s)_+)^{*(rS)}, (\bar{L}_S)^r], \\ \bar{D}_s(L_K)^r &= -(-1)^{srKS} [(((\bar{L}_S)^s)_-)^{*(rK)}, (L_K)^r], \\ \bar{D}_s(\bar{L}_S)^r &= (-1)^{srS} [(((\bar{L}_S)^s)_+)^{*(rS)}, (\bar{L}_S)^r], \quad s \in \mathbb{N}. \end{aligned} \quad (4.4)$$

The Z_2 -parity of two kinds of derivatives is defined as

$$d_{D_s} = sK \bmod 2, \quad d_{\bar{D}_s} = sS \bmod 2.$$

The Lax equations generate a non-Abelian super algebra of flows of the 2D fermionic (K, S) - Toda lattice hierarchy

$$[D_s, D_p] = (1 - (-1)^{spK})D_{s+p}, \quad [\bar{D}_s, \bar{D}_p] = (1 - (-1)^{spS})\bar{D}_{s+p}, \quad [D_s, \bar{D}_p] = 0.$$

If we introduce the notation $v_{S0}(x) = \alpha(x)$, $v_{S1}(x) = \rho(x)$, $u_{K1}(x) = \gamma(x)$, $u_{K2}(x) = \beta(x)$ and consider eqs. (4.4) at $K = S = 2$, $r = s = 1$. One obtains

$$\begin{aligned} D_1\alpha(x) &= \alpha(x)(\beta(x) - \beta(x - 2\epsilon)), & \bar{D}_1\gamma(x) &= \rho(x)u_0(x - \epsilon) - \rho(x + 2\epsilon)u_0(x), \\ \bar{D}_1\beta(x) &= \alpha(x)u_0(x - 2\epsilon) - \alpha(x + 2\epsilon)u_0(x) - \gamma(x)\rho(x + \epsilon) - \gamma(x - \epsilon)\rho(x), \\ D_1\rho(x) &= \rho(x)(\beta(x) - \beta(x - \epsilon)) + \alpha(x + \epsilon)\gamma(x) - \alpha(x)\gamma(x - 2\epsilon), & \bar{D}_1u_{K0}(x) &= 0. \end{aligned} \quad (4.5)$$

It is easy to check that after a reduction $u_{K0}(x) = 1$ eqs.(4.5) will become the 2D generalized fermionic Toda lattice equations as

$$\begin{aligned} D_1\alpha(x) &= \alpha(x)(\beta(x) - \beta(x - 2\epsilon)), & \bar{D}_1\gamma(x) &= \rho(x) - \rho(x + 2\epsilon), \\ \bar{D}_1\beta(x) &= \alpha(x) - \alpha(x + 2\epsilon) - \gamma(x)\rho(x + \epsilon) - \gamma(x - \epsilon)\rho(x), \\ D_1\rho(x) &= \rho(x)(\beta(x) - \beta(x - \epsilon)) + \alpha(x + \epsilon)\gamma(x) - \alpha(x)\gamma(x - 2\epsilon). \end{aligned} \quad (4.6)$$

4.1 2D fermionic $(2N, 2M)$ -Toda hierarchy

When $K = 2N, S = 2M$, then the Lax operators L_{2N}, \bar{L}_{2M} will be defined to satisfy the following specific flow equations of the 2D fermionic $(2N, 2M)$ -Toda hierarchy

$$D_{2nN}L_{2N} = -[(L_{2N}^n)_-, L_{2N}], \quad D_{2nN}\bar{L}_{2M} = [(L_{2N}^n)_+, \bar{L}_{2M}], \quad (4.7)$$

$$\bar{D}_{2nM}L_{2N} = -[(\bar{L}_{2M}^n)_-, L_{2N}], \quad \bar{D}_{2nM}\bar{L}_{2M} = [(\bar{L}_{2M}^n)_+, \bar{L}_{2M}]. \quad (4.8)$$

In the next section, we will study on the additional symmetries of this 2D fermionic $(2N, 2M)$ -Toda hierarchy.

5 Additional symmetry of 2D fermionic $(2N, 2M)$ -Toda hierarchy

In this section, we will consider the construction of the flows of additional symmetries of the 2D fermionic $(2N, 2M)$ -Toda hierarchy.

Firstly, we introduce Orlov-Schulman operators as following

$$M_N = W\Gamma_N W^{-1}, \quad \bar{M}_M = \bar{W}\bar{\Gamma}_M \bar{W}^{-1}, \quad (5.1)$$

$$\Gamma_N = \frac{x}{2N\epsilon}\Lambda^{-2nN} + \sum_{n \geq 0} n\Lambda^{2N(n-1)}t_{2nN}, \quad \bar{\Gamma}_M = \frac{-x}{2M\epsilon}\Lambda^{2nM} - \sum_{n \geq 0} n\Lambda^{-2M(n-1)}\bar{t}_{2nM}. \quad (5.2)$$

Then one can prove the Lax operator L_{2N}, \bar{L}_{2M} and Orlov-Schulman operators M_N, \bar{M}_M all have the even Grassmann parity and satisfy the following theorem.

Proposition 5.1. *The Lax operators L_{2N}, \bar{L}_{2M} and Orlov-Schulman operators M_N, \bar{M}_M of the fermionic $(2N, 2M)$ -Toda hierarchy satisfy the following*

$$\begin{aligned} [L_{2N}, M_N] &= 1, [\bar{L}_{2M}, \bar{M}_M] = 1, \\ D_{2nN}(M_N)^r &= [((L_{2N})^n)_+, (M_N)^r], \\ D_{2nN}(\bar{M}_M)^r &= [((L_{2N})^n)_+, (\bar{M}_M)^r], \\ \bar{D}_{2nM}(M_N)^r &= -[((\bar{L}_{2M})^n)_-, (M_N)^r], \\ \bar{D}_{2nM}(\bar{M}_M)^r &= -[((\bar{L}_{2M})^n)_-, (\bar{M}_M)^r], \quad n \in \mathbb{N}. \end{aligned} \quad (5.3)$$

Proof. One can prove the proposition by dressing the following several commutative Lie brackets

$$\begin{aligned} &[D_{t_{2nN}} - \Lambda^{2nN}, \Gamma_N] \\ &= [D_{t_{2nN}} - \Lambda^{2nN}, \frac{x}{2N\epsilon} \Lambda^{-2N} + \sum_{n \geq 0} n \Lambda^{2N(n-1)} t_n] \\ &= 0, \end{aligned}$$

$$[D_{t_{2nN}}, \bar{\Gamma}_M] = [D_{t_{2nN}}, \frac{-x}{2M\epsilon} \Lambda^{2nM}] = 0.$$

The other identities can be proved in similar ways. \square

For the additional symmetries of the fermionic $(2N, 2M)$ -Toda hierarchy, we introduce additional independent variables $t_{m,l}, \bar{t}_{m,l}$ and define the actions of the additional flows on the wave operators as

$$\epsilon D_{t_{m,l}} W = -(M_N^m L_{2N}^l)_- W, \quad \epsilon D_{\bar{t}_{m,l}} \bar{W} = (\bar{M}_M^m \bar{L}_{2M}^l)_+ \bar{W}, \quad (5.5)$$

$$\epsilon D_{t_{m,l}} \bar{W} = (M_N^m L_{2N}^l)_+ \bar{W}, \quad \epsilon D_{\bar{t}_{m,l}} W = -(\bar{M}_M^m \bar{L}_{2M}^l)_- W, \quad (5.6)$$

where $m \geq 0, l \geq 0$.

By considering the well-known w_∞ algebra with algebraic coefficients $C_{\alpha\beta}^{(ps)(ab)}$ as

$$[z^s \partial^p, z^b \partial^a] = \sum_{\alpha\beta} C_{\alpha\beta}^{(ps)(ab)} z^\beta \partial_\alpha,$$

we can prove the above additional flows are in fact symmetries of the fermionic $(2N, 2M)$ -Toda hierarchy whose structure is a $w_\infty \times w_\infty$ algebra as in the following theorem.

Theorem 5.2. *The additional flows $D_{t_{m,l}}, D_{\bar{t}_{m,l}}$ of the fermionic $(2N, 2M)$ -Toda hierarchy form a $w_\infty \times w_\infty$ Lie algebra with the following relation*

$$[D_{t_{m,l}}, D_{t_{n,k}}] = \sum_{\alpha\beta} C_{\alpha\beta}^{(ml)(nk)} D_{t_{\alpha,\beta}}, \quad (5.7)$$

$$[D_{\bar{t}_{m,l}}, D_{\bar{t}_{n,k}}] = \sum_{\alpha\beta} C_{\alpha\beta}^{(ml)(nk)} D_{\bar{t}_{\alpha,\beta}}, \quad (5.8)$$

$$[D_{t_{m,l}}, D_{\bar{t}_{n,k}}] = 0, \quad (5.9)$$

which holds in the sense of acting on W, \bar{W} or L_{2N}, \bar{L}_{2M} and $m, n, l, k \geq 0$.

Proof. The proof can be proved similarly as the additional symmetry of two dimensional Toda hierarchy in [30]. Here we skip the proof. \square

6 Extended fermionic $(2N, 2M)$ -Toda hierarchy

In this section, we consider the reduction of the 2D fermionic $(2N, 2M)$ -Toda lattice hierarchy for even values of $(2N, 2M)$ to the 1D space with additional logarithmic flows.

For even $(2N, 2M)$, one can impose the reduction constraint on the Lax operators (4.1) as follows:

$$L^{2N} = \bar{L}^{2M} \equiv \mathcal{L}, \quad (6.1)$$

which leads to the following explicit form for the reduced Lax operator

$$\mathcal{L} = \sum_{k=0}^{2N+2M} u_k(x) \Lambda^{2N-k}. \quad (6.2)$$

One can find $\mathcal{L}_*^k = \mathcal{L}^k$, $k \in \mathbb{Z}_+$. The Lax equation of the one dimensional fermionic $(2N, 2M)$ -Toda hierarchy on the reduced Lax operator

$$\epsilon D_{t_s} \mathcal{L} = [(\frac{\mathcal{L}^{s+1}}{(s+1)!})_+, \mathcal{L}], \quad (6.3)$$

where the Lax operator \mathcal{L} has the following dressing structure

$$\mathcal{L} = W \Lambda^{2N} W^{-1} = \bar{W} \Lambda^{-2M} \bar{W}^{-1}. \quad (6.4)$$

Similarly as the [8, 9], we define the following logarithmic operator

$$\log_+ \mathcal{L} = 2N W \circ \epsilon \partial \circ W^{-1}, \quad (6.5)$$

$$\log_- \mathcal{L} = -2M \bar{W} \circ \epsilon \partial \circ \bar{W}^{-1}, \quad (6.6)$$

where ∂ is the derivative about spatial variable x . Combining these above logarithm operators together can derive the following important logarithmic operator

$$\log \mathcal{L} := \frac{1}{4} \left(\frac{1}{N} \log_+ \mathcal{L} + \frac{1}{M} \log_- \mathcal{L} \right) = \sum_{i=-\infty}^{+\infty} w_i \Lambda^i, \quad (6.7)$$

which will generate a series of flow equations which contain the spatial flow in later defined Lax equations. Here the logarithmic Lax operator $\log \mathcal{L}$ has the even Grassmann parity.

Under the reduction $u_0(x) = 1$, one can derive the 1D $(2, 2)$ -Toda lattice hierarchy in [26]. In this case, the representation (6.3) will be given by the Lax operator

$$L_{2,2} = \Lambda^2 + \gamma(x) \Lambda + \beta(x) + \rho(x) \Lambda^{-1} + \alpha(x) \Lambda^{-2}.$$

As a consequence of eq. (6.3), we have the D_0 flow equations as

$$\begin{aligned} D_0 \alpha(x) &= \alpha(x)(\beta(x) - \beta(x - 2\epsilon)), \\ D_0 \rho(x) &= \rho(x)(\beta(x) - \beta(x - \epsilon)) + \alpha(x + \epsilon)\gamma(x) - \alpha(x)\gamma(x - 2\epsilon), \\ D_0 \gamma(x) &= \rho(x + 2\epsilon) - \rho(x), \\ D_0 \beta(x) &= \alpha(x + 2\epsilon) - \alpha(x) + \gamma(x)\rho(x + \epsilon) + \gamma(x - \epsilon)\rho(x). \end{aligned}$$

6.1 Lax equations of extended fermionic $(2N, 2M)$ -Toda hierarchy

In this section we will introduce the Lax equations of extended fermionic $(2N, 2M)$ -Toda hierarchy. Let us first introduce some convenient notations.

Definition 6.1. *The operators B_j, G_j are defined as follows*

$$B_j := \frac{\mathcal{L}^{j+1}}{(j+1)!}, \quad G_j := \frac{2\mathcal{L}^j}{j!}(\log \mathcal{L} - c_j), \quad c_j = \sum_{i=1}^j \frac{1}{i}, \quad j \geq 0. \quad (6.8)$$

Now we give the definition of the extended fermionic $(2N, 2M)$ -Toda hierarchy.

Definition 6.2. *The extended fermionic $(2N, 2M)$ -Toda hierarchy is a hierarchy in which the dressing operators W, \bar{W} satisfy the following Sato equations*

$$\epsilon D_{t_j} W = -(B_j)_- W, \quad \epsilon D_{t_j} \bar{W} = (B_j)_+ \bar{W}, \quad (6.9)$$

$$\epsilon D_{y_j} W = -(G_j)_- W, \quad \epsilon D_{y_j} \bar{W} = (G_j)_+ \bar{W}. \quad (6.10)$$

From the previous proposition we derive the following Lax equations for the Lax operators.

Proposition 6.3. *The Lax equations of the extended fermionic $(2N, 2M)$ -Toda hierarchy are as follows*

$$\epsilon D_{t_j} \mathcal{L} = [(B_j)_+, \mathcal{L}], \quad \epsilon D_{y_j} \mathcal{L} = [(G_j)_+, \mathcal{L}], \quad \epsilon D_{t_j} \log \mathcal{L} = [(B_j)_+, \log \mathcal{L}], \quad (6.11)$$

$$\epsilon D_{y_j} \log \mathcal{L} = [-(G_j)_-, \log_+ \mathcal{L}] + [(G_j)_+, \log_- \mathcal{L}]. \quad (6.12)$$

7 Additional symmetry and Block algebra

In this section, we will construct of the flows of additional symmetries which form the well-known Block algebra. With the dressing operators given M, \bar{M} , we introduce Orlov-Schulman operators as following

$$M = W\Gamma W^{-1}, \quad \bar{M} = \bar{W}\bar{\Gamma}\bar{W}^{-1}, \quad (7.1)$$

$$\Gamma = \frac{x}{2N\epsilon} \Lambda^{-2N} + \sum_{n \geq 0} \frac{\Lambda^{2Nn}}{n} t_n + \sum_{n \geq 0} \frac{2}{(n-1)!} \Lambda^{2N(n-1)} (\epsilon \partial - c_{n-1}) y_n, \quad (7.2)$$

$$\bar{\Gamma} = \frac{-x}{2M\epsilon} \Lambda^{2M} + \sum_{n \geq 0} \frac{2}{(n-1)!} \Lambda^{2M(1-n)} (-\epsilon \partial - c_n) y_n. \quad (7.3)$$

Then one can prove the Lax operator \mathcal{L} and Orlov-Schulman operators M, \bar{M} all have the even Grassmann parity and satisfy the following theorem.

Proposition 7.1. *The Lax operator \mathcal{L} and Orlov-Schulman operators M, \bar{M} of the extended fermionic $(2N, 2M)$ -Toda hierarchy satisfy the following*

$$[\mathcal{L}, M] = 1, [\mathcal{L}, \bar{M}] = 1, [\log_+ \mathcal{L}, M] = W\Lambda^{-1}W^{-1}, [\log_- \mathcal{L}, \bar{M}] = \bar{W}\Lambda\bar{W}^{-1}, \quad (7.4)$$

$$\epsilon D_{t_k} M^m \mathcal{L}^k = [(B_k)_+, M^m \mathcal{L}^k], \quad \epsilon D_{t_k} \bar{M}^m \mathcal{L}^k = [(B_k)_+, \bar{M}^m \mathcal{L}^k], \quad (7.5)$$

$$\epsilon D_{y_j} M^m \mathcal{L}^k = [\frac{\mathcal{L}^j}{j!}(\log_+ \mathcal{L} - c_j) - (G_j)_-, M^m \mathcal{L}^k], \quad \epsilon D_{y_j} \bar{M}^m \mathcal{L}^k = [-\frac{\mathcal{L}^j}{j!}(\log_- \mathcal{L} - c_j) + (G_j)_+, \bar{M}^m \mathcal{L}^k]. \quad (7.6)$$

Proof. One can prove the proposition by dressing the following several commutative Lie brackets

$$\begin{aligned}
& [D_{t_n} - \frac{\Lambda^{2(n+1)N}}{(n+1)!}, \Gamma\} \\
&= [D_{t_n} - \frac{\Lambda^{2(n+1)N}}{(n+1)!}, \frac{x}{2N\epsilon} \Lambda^{-2N} + \sum_{n \geq 0} \frac{\Lambda^{2Nn}}{n} t_n + \sum_{n \geq 0} \frac{2}{(n-1)!} \Lambda^{2N(n-1)} (\epsilon \partial - c_{n-1}) y_n\} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
& [D_{y_n} - \frac{2}{n!} \Lambda^{2Nn} (\epsilon \partial - c_n), \Gamma\} \\
&= [D_{y_n} - \frac{2}{n!} \Lambda^{2Nn} (\epsilon \partial - c_n), \frac{x}{2N\epsilon} \Lambda^{-2N} + \sum_{n \geq 0} \frac{\Lambda^{2Nn}}{n} t_n + \sum_{n \geq 0} \frac{2}{(n-1)!} \Lambda^{2N(n-1)} (\epsilon \partial - c_{n-1}) y_n\} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
& [D_{y_n} + \frac{2}{n!} \Lambda^{-2Mn} (-\epsilon \partial - c_n), \bar{\Gamma}\} \\
&= [D_{y_n} + \frac{2}{n!} \Lambda^{-2Mn} (-\epsilon \partial - c_n), \frac{-x}{2M\epsilon} \Lambda^{2M} + \sum_{n \geq 0} \frac{2}{(n-1)!} \Lambda^{2M(1-n)} (-\epsilon \partial - c_n) y_n\} \\
&= 0.
\end{aligned}$$

The other identities can be proved by similar dressing methods which will be skipped here. \square

We are now to define the additional flows, and then to prove that they are symmetries, which are called additional symmetries of the extended fermionic $(2N, 2M)$ -Toda hierarchy. We introduce additional independent variables $t_{m,l}^*$ and define the actions of the additional flows on the wave operators as

$$\epsilon D_{t_{m,l}^*} W = - \left((M - \bar{M})^m \mathcal{L}^l \right)_- W, \quad \epsilon D_{t_{m,l}^*} \bar{W} = \left((M - \bar{M})^m \mathcal{L}^l \right)_+ \bar{W}, \quad (7.7)$$

where $m \geq 0, l \geq 0$. The following theorem shows that the definition (7.7) is compatible with reduction condition (6.4) of the extended fermionic $(2N, 2M)$ -Toda hierarchy.

Proposition 7.2. *The additional flows (7.7) preserve reduction condition (4.1) or (6.4) of the extended fermionic $(2N, 2M)$ -Toda hierarchy.*

Proof. By performing the derivative on \mathcal{L} dressed by W and using the additional flow about W in (7.7), we get

$$\begin{aligned}
(\epsilon D_{t_{m,l}^*} \mathcal{L}) &= (\epsilon D_{t_{m,l}^*} W) \Lambda^{2N} W^{-1} + W \Lambda^{2N} (\epsilon D_{t_{m,l}^*} W^{-1}) \\
&= -((M - \bar{M})^m \mathcal{L}^l)_- W \Lambda^{2N} W^{-1} - W \Lambda^{2N} W^{-1} (\epsilon D_{t_{m,l}^*} W) W^{-1} \\
&= -((M - \bar{M})^m \mathcal{L}^l)_- \mathcal{L} + \mathcal{L} ((M - \bar{M})^m \mathcal{L}^l)_- \\
&= -[((M - \bar{M})^m \mathcal{L}^l)_-, \mathcal{L}].
\end{aligned}$$

Similarly, we perform the derivative on \mathcal{L} dressed by \bar{W} and use the additional flow about \bar{W} in (7.7) to get the following

$$\begin{aligned}
(\epsilon D_{t_{m,l}}^* \mathcal{L}) &= (\epsilon D_{t_{m,l}}^* \bar{W}) \Lambda^{-2M} \bar{W}^{-1} + \bar{W} \Lambda^{-2M} (\epsilon D_{t_{m,l}} \bar{W}^{-1}) \\
&= ((M - \bar{M})^m \mathcal{L}^l)_+ \bar{W} \Lambda^{-2M} \bar{W}^{-1} - \bar{W} \Lambda^{-2M} \bar{W}^{-1} (\epsilon D_{t_{m,l}}^* \bar{W}) \bar{W}^{-1} \\
&= ((M - \bar{M})^m \mathcal{L}^l)_+ \mathcal{L} - \mathcal{L} ((M - \bar{M})^m \mathcal{L}^l)_+ \\
&= [((M - \bar{M})^m \mathcal{L}^l)_+, \mathcal{L}].
\end{aligned}$$

Because

$$[M - \bar{M}, \mathcal{L}] = 0, \quad (7.8)$$

therefore

$$\epsilon D_{t_{m,l}}^* \mathcal{L} = [-((M - \bar{M})^m \mathcal{L}^l)_-, \mathcal{L}] = [((M - \bar{M})^m \mathcal{L}^l)_+, \mathcal{L}], \quad (7.9)$$

which gives the compatibility of additional flow of extended fermionic $(2N, 2M)$ -Toda hierarchy with reduction condition (6.4). \square

Similarly, by performing the derivative on M given in (7.1), there exists a similar derivative as $\epsilon D_{t_{m,l}}^* \mathcal{L}$, i.e.,

$$\begin{aligned}
(\epsilon D_{t_{m,l}}^* M) &= (\epsilon D_{t_{m,l}}^* W) \Gamma W^{-1} + W \Gamma (\epsilon D_{t_{m,l}} W^{-1}) \\
&= -((M - \bar{M})^m \mathcal{L}^l)_- W \Gamma W^{-1} - W \Gamma W^{-1} (\epsilon D_{t_{m,l}}^* W) W^{-1} \\
&= -((M - \bar{M})^m \mathcal{L}^l)_- M + M ((M - \bar{M})^m \mathcal{L}^l)_- \\
&= -[((M - \bar{M})^m \mathcal{L}^l)_-, M],
\end{aligned}$$

where the fact that Γ does not depend on the additional variables $t_{m,l}^*$ has been used. Then we can take derivatives on the dressing structure of \bar{M} to get the additional derivatives act on M, \bar{M} as

$$\epsilon D_{t_{m,l}}^* M = [-((M - \bar{M})^m \mathcal{L}^l)_-, M], \quad (7.10)$$

$$\epsilon D_{t_{m,l}}^* \bar{M} = [((M - \bar{M})^m \mathcal{L}^l)_+, \bar{M}]. \quad (7.11)$$

By above results, the following corollary can be easily got.

Corollary 7.3. *For $n, k, m, l \geq 0$, the following identities hold true*

$$\epsilon D_{t_{m,l}}^* M^n \mathcal{L}^k = -[((M - \bar{M})^m \mathcal{L}^l)_-, M^n \mathcal{L}^k], \quad \epsilon D_{t_{m,l}}^* \bar{M}^n \mathcal{L}^k = [((M - \bar{M})^m \mathcal{L}^l)_+, \bar{M}^n \mathcal{L}^k]. \quad (7.12)$$

With Corollary 7.3, the following theorem can be proved.

Theorem 7.4. *The additional flows $D_{t_{m,l}}^*$ commute with the extended fermionic $(2N, 2M)$ -Toda hierarchy flows D_{t_n}, D_{y_n} , i.e.,*

$$[D_{t_{m,l}}^*, D_{t_n}] \Phi = 0, \quad [D_{t_{m,l}}^*, D_{y_n}] \Phi = 0, \quad (7.13)$$

where Φ can be W, \bar{W} or \mathcal{L} , $n \geq 0$.

Proof. According to the definition,

$$[D_{t_{m,l}}^*, D_{t_n}]W = D_{t_{m,l}}^*(D_{t_n}W) - D_{t_n}(D_{t_{m,l}}^*W),$$

and using the actions of the additional flows and the fermionic $(2N, 2M)$ -Toda flows on W , we have

$$\begin{aligned} \epsilon^2[D_{t_{m,l}}^*, D_{t_n}]W &= -\epsilon D_{t_{m,l}}^*((B_n)_-W) + \epsilon D_{t_n}(((M - \bar{M})^m \mathcal{L}^l)_-W) \\ &= -\epsilon(D_{t_{m,l}}^*B_n)_-W - \epsilon(B_n)_-(D_{t_{m,l}}^*W) \\ &\quad + \epsilon[D_{t_n}((M - \bar{M})^m \mathcal{L}^l)]_-W + \epsilon((M - \bar{M})^m \mathcal{L}^l)_-(D_{t_n}W). \end{aligned}$$

Using (7.7) and Proposition 7.1, it equals

$$\begin{aligned} \epsilon^2[D_{t_{m,l}}^*, D_{t_n}]W &= [((M - \bar{M})^m \mathcal{L}^l)_-, B_n]_-W + (B_n)_-((M - \bar{M})^m \mathcal{L}^l)_-W \\ &\quad + [(B_n)_+, (M - \bar{M})^m \mathcal{L}^l]_-W - ((M - \bar{M})^m \mathcal{L}^l)_-(B_n)_-W \\ &= [((M - \bar{M})^m \mathcal{L}^l)_-, B_n]_-W - [(M - \bar{M})^m \mathcal{L}^l, (B_n)_+]_-W \\ &\quad + [(B_n)_-, ((M - \bar{M})^m \mathcal{L}^l)_+]W \\ &= 0. \end{aligned}$$

Similarly, using (7.7) and Proposition 7.1, we can prove the additional flows commute with flows ∂_{t_n} in the sense of acting on W . Here we also give the proof for commutativity of additional symmetries with the extended flow ∂_{y_n} . To be a little different from the proof above, we let the Lie bracket act on \bar{W} ,

$$\begin{aligned} \epsilon^2[D_{t_{m,l}}^*, D_{y_n}]\bar{W} &= \epsilon D_{t_{m,l}}^*(G_n)_+\bar{W} - \epsilon D_{y_n}(((M - \bar{M})^m \mathcal{L}^l)_+\bar{W}) \\ &= \epsilon D_{t_{m,l}}^*(G_n)_+\bar{W} + \epsilon(G_n)_+(D_{t_{m,l}}^*\bar{W}) \\ &\quad - \epsilon(D_{y_n}((M - \bar{M})^m \mathcal{L}^l))_+\bar{W} - \epsilon((M - \bar{M})^m \mathcal{L}^l)_+(D_{y_n}\bar{W}), \end{aligned}$$

which further leads to

$$\begin{aligned} \epsilon^2[D_{t_{m,l}}^*, D_{y_n}]\bar{W} &= [((M - \bar{M})^m \mathcal{L}^l)_+, \frac{\mathcal{L}^n}{n!}(\log_- \mathcal{L} - c_n)]_+\bar{W} \\ &\quad - [((M - \bar{M})^m \mathcal{L}^l)_-, \frac{\mathcal{L}^n}{n!}(\log_+ \mathcal{L} - c_n)]_+\bar{W} \\ &\quad + (G_n)_+((M - \bar{M})^m \mathcal{L}^l)_+\bar{W} - ((M - \bar{M})^m \mathcal{L}^l)_+(G_n)_+\bar{W} \\ &\quad - \left[\left(\frac{\mathcal{L}^n}{n!}(\log_+ \mathcal{L} - c_n) \right)_+ - \left(\frac{\mathcal{L}^n}{n!}(\log_- \mathcal{L} - c_n) \right)_- , (M - \bar{M})^m \mathcal{L}^l \right]_+\bar{W} \\ &= [((M - \bar{M})^m \mathcal{L}^l)_+, \left(\frac{\mathcal{L}^n}{n!}(\log_- \mathcal{L} - c_n) \right)_+]_+\bar{W} \\ &\quad - [((M - \bar{M})^m \mathcal{L}^l)_-, \frac{\mathcal{L}^n}{n!}(\log_+ \mathcal{L} - c_n)]_+\bar{W} \\ &\quad + (G_n)_+((M - \bar{M})^m \mathcal{L}^l)_+\bar{W} - ((M - \bar{M})^m \mathcal{L}^l)_+(G_n)_+\bar{W} \\ &\quad + [(M - \bar{M})^m \mathcal{L}^l, \left(\frac{\mathcal{L}^n}{n!}(\log_+ \mathcal{L} - c_n) \right)_+]_+\bar{W} \\ &= 0. \end{aligned}$$

The other cases in the theorem can be proved in similar ways. □

The commutative property in Theorem 7.4 means that additional flows are symmetries of the extended fermionic $(2N, 2M)$ -Toda hierarchy. Since they are symmetries, we will consider the algebraic structure among these additional symmetries which is included in the following important theorem.

Theorem 7.5. *The additional flows $\epsilon D_{t_{m,l}^*}$ of the extended fermionic $(2N, 2M)$ -Toda hierarchy form a Block type Lie algebra with the following relation*

$$[\epsilon D_{t_{m,l}^*}, \epsilon D_{t_{n,k}^*}] = (km - nl)\epsilon D_{t_{m+n-1,k+l-1}^*}, \quad (7.14)$$

which holds in the sense of acting on W , \bar{W} or \mathcal{L} and $m, n, l, k \geq 0$.

Proof. By using the additional flows, we get

$$\begin{aligned} \epsilon^2 [D_{t_{m,l}^*}, D_{t_{n,k}^*}] W &= \epsilon^2 D_{t_{m,l}^*} (D_{t_{n,k}^*} W) - \epsilon^2 D_{t_{n,k}^*} (D_{t_{m,l}^*} W) \\ &= -\epsilon D_{t_{m,l}^*} ((M - \bar{M})^n \mathcal{L}^k)_- W + \epsilon D_{t_{n,k}^*} ((M - \bar{M})^m \mathcal{L}^l)_- W \\ &= -\epsilon (D_{t_{m,l}^*} (M - \bar{M})^n \mathcal{L}^k)_- W - \epsilon ((M - \bar{M})^n \mathcal{L}^k)_- (D_{t_{m,l}^*} W) \\ &\quad + \epsilon (D_{t_{n,k}^*} (M - \bar{M})^m \mathcal{L}^l)_- W + \epsilon ((M - \bar{M})^m \mathcal{L}^l)_- (D_{t_{n,k}^*} W). \end{aligned}$$

We further get

$$\begin{aligned} &\epsilon^2 [D_{t_{m,l}^*}, D_{t_{n,k}^*}] W \\ &= -\epsilon \left[\sum_{p=0}^{n-1} (M - \bar{M})^p (D_{t_{m,l}^*} (M - \bar{M})) (M - \bar{M})^{n-p-1} \mathcal{L}^k + (M - \bar{M})^n (D_{t_{m,l}^*} \mathcal{L}^k) \right]_- W \\ &\quad - ((M - \bar{M})^n \mathcal{L}^k)_- (D_{t_{m,l}^*} W) \\ &\quad + \epsilon \left[\sum_{p=0}^{m-1} (M - \bar{M})^p (D_{t_{n,k}^*} (M - \bar{M})) (M - \bar{M})^{m-p-1} \mathcal{L}^l + (M - \bar{M})^m (D_{t_{n,k}^*} \mathcal{L}^l) \right]_- W \\ &\quad + \epsilon ((M - \bar{M})^m \mathcal{L}^l)_- (D_{t_{n,k}^*} W) \\ &= [(nl - km)(M - \bar{M})^{m+n-1} \mathcal{L}^{k+l-1}]_- W \\ &= (km - nl)\epsilon D_{t_{m+n-1,k+l-1}^*} W. \end{aligned}$$

Similarly the same results on acting on \bar{W} and \mathcal{L} are as follows

$$\begin{aligned} \epsilon^2 [D_{t_{m,l}^*}, D_{t_{n,k}^*}] \bar{W} &= ((km - nl)(M - \bar{M})^{m+n-1} \mathcal{L}^{k+l-1})_+ \bar{W} \\ &= (km - nl)\epsilon D_{t_{m+n-1,k+l-1}^*} \bar{W}, \\ \epsilon^2 [D_{t_{m,l}^*}, D_{t_{n,k}^*}] \mathcal{L} &= \epsilon^2 D_{t_{m,l}^*} (D_{t_{n,k}^*} \mathcal{L}) - \epsilon^2 D_{t_{n,k}^*} (D_{t_{m,l}^*} \mathcal{L}) \\ &= [((nl - km)(M - \bar{M})^{m+n-1} \mathcal{L}^{k+l-1})_-, \mathcal{L}] \\ &= (km - nl)\epsilon D_{t_{m+n-1,k+l-1}^*} \mathcal{L}. \end{aligned}$$

□

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